

# Extendibility , monodromy and local triviality for topological groupoids \*

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February 1, 2008

## Abstract

A groupoid is a small category in which each morphism has an inverse. A topological groupoid is a groupoid in which both sets of objects and morphisms have topologies such that all groupoid structure maps are continuous. The notion of monodromy groupoid of a topological groupoid generalises those of fundamental groupoid and universal covering. It was earlier proved that the monodromy of a locally sectionable topological groupoid has a topological groupoid structure satisfying some properties. In this paper a similar problem is studied for compatible locally trivial topological groupoids.

## Introduction

Let  $G$  be a topological groupoid and  $W$  an open subset of  $G$  containing all the identities. Then the monodromy groupoid  $M(G, W)$  of  $G$  is defined as in Definition 1.2, which is due to Pradines [12]. In [11] (see also [5] and [6] for Lie groupoid case ) in the case where  $G$  is locally sectionable it was proved that the groupoid  $M(G, W)$  may be given a topology making it a topological groupoid such that each star  $M(G, W)_x$  is a universal covering of  $G_x$  and  $M(G, W)$  has a universal property on the globalisation of continuous pregroupoid morphisms to topological groupoid morphisms.

If  $X$  is a topological space then  $G = X \times X$  is a topological groupoid with the groupoid multiplication  $(x, y)(y, z) = (x, z)$ . If  $X$  is a path connected topological space which has a universal covering then the monodromy groupoid of the topological groupoid  $G = X \times X$  for a suitable open neighbourhood  $W$  is the fundamental groupoid  $\pi_1(X)$ .

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\*KEYWORDS: Extendibility, Locally triviality, topological groupoid, monodromy groupoid: 1991 AMS Classification: 22A05, 55M99, 55R15

If  $G$  is a topological group, which can be thought as a topological groupoid with only one object, such that it has a universal covering, then for a suitable open subset  $W$  of  $G$  the monodromy groupoid  $M(G, W)$  is just the universal covering of  $G$ .

Thus the concept of the monodromy groupoid generalises the notions of the fundamental groupoid as topological groupoid and the universal covering.

In [10] the monodromy groupoid denoted by  $\Pi G$  of a topological groupoid  $G$  in which each star  $G_x$  has a universal covering is constructed directly from the universal coverings of  $G_x$ 's, and then  $\Pi G$  is given a topology under some conditions studying in terms of principal bundles.

In this paper, we define compatible locally trivial groupoid and as similar to the fundamental groupoid case studied in [3] prove that the monodromy groupoid of a compatible locally trivial groupoid can be topologised with an appropriate topology.

## 1 Groupoids and topological groupoids

A *groupoid*  $G$  on  $O_G$  is a small category in which each morphism is an isomorphism. Thus  $G$  has a set of morphisms, which we call just *elements* of  $G$ , a set  $O_G$  of *objects* together with functions  $\alpha, \beta: G \rightarrow O_G$ ,  $\epsilon: O_G \rightarrow G$  such that  $\alpha\epsilon = \beta\epsilon = 1_{O_G}$ , the identity map. The functions  $\alpha, \beta$  are called *initial* and *final* point maps respectively. If  $a, b \in G$  and  $\beta(a) = \alpha(b)$ , then the *product* or *composite*  $ab$  exists such that  $\alpha(ab) = \alpha(a)$  and  $\beta(ab) = \beta(b)$ . Further, this composite is associative, for  $x \in O_G$  the element  $\epsilon(x)$  denoted by  $1_x$  acts as the identity, and each element  $a$  has an inverse  $a^{-1}$  such that  $\alpha(a^{-1}) = \beta(a)$ ,  $\beta(a^{-1}) = \alpha(a)$ ,  $aa^{-1} = (\epsilon\alpha)(a)$ ,  $a^{-1}a = (\epsilon\beta)(a)$ . The map  $G \rightarrow G$ ,  $a \mapsto a^{-1}$  is called the *inversion*.

In a groupoid  $G$  for  $x, y \in O_G$  we write  $G(x, y)$  for the set of all morphisms with initial point  $x$  and final point  $y$ . For  $x \in O_G$  we denote the star  $\{a \in G: \alpha(a) = x\}$  of  $x$  by  $G_x$  and the *costar*  $\{a \in G: \beta(a) = x\}$  of  $x$  by  $G^x$ . In  $G$  the set  $O_G$  is mapped bijectively to the set of identities by  $\epsilon: O_G \rightarrow G$ . So we sometimes write  $O_G$  for the set of identities. Let  $G$  be a groupoid and  $W$  a subset of  $G$  such that  $W \subseteq O_G$ . We say  $G$  is *generated* by  $W$  if each element of  $G$  is written as a union of some elements in  $W$ .

Let  $G$  be a groupoid. A *subgroupoid* of  $G$  is a pair of subsets  $H \subseteq G$  and  $O_H \subseteq O_G$  such that  $\alpha(H) \subseteq O_H$ ,  $\beta(H) \subseteq O_H$ ,  $1_x \in H$  for each  $x \in O_H$  and  $H$  is closed under the partial multiplication and inversion in  $G$ .

A *morphism* of groupoids  $H$  and  $G$  is a functor, that is, it consists of a pair of functions  $f: H \rightarrow G$  and  $O_f: O_H \rightarrow O_G$  preserving all the structures.

**Definition 1.1** ([10]) A *topological groupoid* is a groupoid  $G$  on  $O_G$  together with topologies on  $G$  and  $O_G$  such that the maps which define the groupoid structure are continuous, namely the initial and final point maps  $\alpha, \beta: G \rightarrow O_G$ , the object inclusion map  $\epsilon: O_G \rightarrow G$ ,  $x \mapsto \epsilon(x)$ , the inversion  $G \rightarrow G$ ,  $a \mapsto a^{-1}$  and the partial multiplication  $G_\alpha \times_\beta G \rightarrow G$ ,  $(a, b) \mapsto ab$ , where the pullback

$$G_\beta \times_\alpha G = \{(a, b) \in G \times G: \beta(a) = \alpha(b)\}$$

has the subspace topology from  $G \times G$ .

A *morphism* of topological groupoids  $f: H \rightarrow G$  is a morphism of groupoids in which both maps  $f: H \rightarrow G$  and  $O_f: O_H \rightarrow O_G$  are continuous.

Note that in this definition the partial multiplication  $G_\beta \times_\alpha G \rightarrow G, (a, b) \mapsto ab$  and the inversion map  $G \rightarrow G, a \mapsto a^{-1}$  are continuous if and only if the map  $\delta: G \times_\alpha G \rightarrow G, (a, b) \mapsto a^{-1}b$ , called the groupoid difference map, is continuous, where the pullback

$$G \times_\alpha G = \{(a, b) \in G \times G : \alpha(a) = \alpha(b)\}$$

has the subspace topology from  $G \times G$ . Again if one of the maps  $\alpha, \beta$  and the inversion are continuous, then the other map is continuous.

Let  $X$  be a topological space. Then  $G = X \times X$  is a topological groupoid on  $X$ . In which each pair  $(x, y)$  is a morphism from  $x$  to  $y$  and the groupoid composite is defined by  $(x, y)(y, z) = (x, z)$ . The inverse of  $(x, y)$  is  $(y, x)$  and the identity at  $1_x$  is the pair  $(x, x)$ .

Note that in a topological groupoid  $G$  for each  $a \in G(x, y)$  right translation  $R_a: G^x \rightarrow G^y, b \mapsto ba$  and left translation  $L_a: G_y \rightarrow G_x, b \mapsto ab$  are homeomorphisms.

A groupoid  $G$  in which each star  $G_x$  has a topology such that for  $a \in G(x, y)$  the right translation  $R_a: G^x \rightarrow G^y, b \mapsto ba$  (and hence the left translation  $L_a: G_y \rightarrow G_x, b \mapsto ab$ ) is homeomorphisms, is called *star topological groupoid*

Let  $G$  be a groupoid and  $W$  a subset of  $G$  such that  $O_G \subseteq W$ . So the set  $W$  inherits a pregroupoid structure from the groupoid  $G$ . That is the source and target maps  $\alpha$  and  $\beta$  restrict to  $W$  and if  $a, b \in W$  and  $\beta a = \alpha b$ , then the composition  $ab$  of  $a$  and  $b$  in  $G$  may or not belong to  $W$ . We follow the method of Brown and Mucuk [5], which generalises work for groups in Douady and Lazard [8].

There is a standard construction  $M(G, W)$  associating to the pregroupoid  $W$  a morphism  $\tilde{i}: W \rightarrow M(G, W)$  to a groupoid  $M(G, W)$  and which is universal for pregroupoid morphism to a groupoid. First we form the free groupoid  $F(W)$  on the graph  $W$  and denote the inclusion  $W \rightarrow F(W)$  by  $a \mapsto [a]$ . Let  $N$  be the normal subgroupoid (Higgins[9], Brown[2]) of  $F(W)$  generated by the elements  $[a][b][ab]^{-1}$  for all  $a, b \in W$  such that  $ab$  is defined and belongs to  $W$ . Then  $M(G, W)$  is defined to be the quotient groupoid  $F(W)/N$ . The composition  $W \rightarrow F(W) \rightarrow M(G, W)$  is written  $\tilde{i}$ , and is the required universal morphism. It is followed that  $\tilde{i}$  is injective.

The map  $\tilde{i}$  has a universal property that if  $f: W \rightarrow H$  is a pregroupoid morphism then there is a morphism of groupoids  $f': M(G, W) \rightarrow H$  such that  $f = f'\tilde{i}$ . In particular the inclusion map  $i: W \rightarrow G$  globalises to a morphism of groupoids  $p: M(G, W) \rightarrow G$  called canonical morphism.

We give this construction as a definition.

**Definition 1.2** Let  $G$  be a topological groupoid and  $W$  an open subset of  $G$  such that  $O_G \subseteq W$ . Let  $F(W)$  be the free groupoid on  $W$  and let  $N$  the normal subgroupoid of  $F(W)$  generated by the elements in the form  $[a][b][ab]^{-1}$  for  $a, b \in W$  such that  $ab$  is defined and  $ab \in W$ . The quotient groupoid  $F(W)/N$  is called *monodromy groupoid* of  $G$  for  $W$  and denoted by  $M(G, W)$ .

## 2 Monodromy groupoids of locally sectionable topological groupoids

In this section we recall some results from [5].

Let  $G$  be a topological groupoid such that each star  $G_x$  has a universal covering. The groupoid  $\Pi G$  as the union of the universal coverings of  $G_x$ 's is defined as follows. As a set,  $\Pi G$  is the union of

the stars  $(\pi_1 G_x)_{1_x}$ . The object set of  $\Pi G$  is the same as that of  $G$ . The function  $\alpha: \Pi G \rightarrow X$  maps all of  $(\pi_1 G_x)_{1_x}$  to  $x$ , while  $\beta: \Pi G \rightarrow X$  is on  $(\pi_1 G_x)_{1_x}$  the composition of the two target maps

$$(\pi_1 G_x)_{1_x} \rightarrow G \rightarrow X.$$

As explained in Mackenzie [10], p.67, there is a multiplication on  $\Pi G$  given by ‘concatenation’, i.e.

$$[a] \circ [b] = [a + a(1)b],$$

where the  $+$  inside the bracket denotes the usual composition of paths. Here  $a$  is assumed to be a path in  $G_x$  from  $1_x$  to  $a(1)$ , where  $\beta(a(1)) = y$ , say, so that  $b$  is a path in  $G_y$ , and for each  $t \in [0, 1]$ , the product  $a(1)b(t)$  is defined in  $G$ , yielding a path  $b(a(1))$  from  $a(1)$  to  $a(1)b(1)$ . It is straightforward to prove that in this way  $\Pi G$  becomes a groupoid, and that the final map of paths induces a morphism of groupoids  $p: \Pi G \rightarrow G$ .

Let  $X$  be a topological space admitting a simply connected cover. A subset  $U$  of  $X$  is called *liftable* if  $U$  is open, path-connected and the inclusion  $U \rightarrow X$  maps each fundamental group of  $U$  trivially. If  $U$  is liftable, and  $q: Y \rightarrow X$  is a covering map, then for any  $y \in Y$  and  $x \in U$  such that  $qy = x$ , there is a unique map  $\hat{i}: U \rightarrow Y$  such that  $\hat{i}x = y$  and  $q\hat{i}$  is the inclusion  $U \rightarrow X$ . This explains the term *liftable*.

**Theorem 2.1** *Suppose that  $G$  is a star connected star topological groupoid and  $W$  is an open neighbourhood of  $O_G$  satisfying the condition*

( $\star$ )  *$W$  is star path-connected and  $W^2$  is contained in a star path-connected neighbourhood  $V$  of  $O_G$  such that for all  $x \in O_G$ ,  $V_x$  is liftable.*

*Then there is an isomorphism of star topological groupoids  $M(G, W) \rightarrow \Pi G$ , and hence the morphism  $p: M(G, W) \rightarrow G$  is a star universal covering map.*

The following definition is due to Ehresmann [7].

**Definition 2.2** Let  $G$  be a groupoid and let  $X = O_G$  be a topological space. An *admissible local section* of  $G$  is a function  $s: U \rightarrow G$  from an open set in  $X$  such that

1.  $\alpha s(x) = x$  for all  $x \in U$ ;
2.  $\beta s(U)$  is open in  $X$ , and
3.  $\beta s$  maps  $U$  homeomorphically to  $\beta s(U)$ .

Let  $W$  be a subset of  $G$  such that  $X \subseteq W$  and let  $W$  have the structure of a topological space. We say that  $(\alpha, \beta, W)$  is *locally sectionable* if for each  $w \in W$  there is an admissible local section  $s: U \rightarrow G$  of  $G$  such that (i)  $s\alpha(w) = w$ , (ii)  $s(U) \subseteq W$  and (iii)  $s$  is smooth as a function from  $U$  to  $W$ . Such an  $s$  is called a smooth *admissible local section*.

The following definition is due to Pradines [12] under the name “*morceau de groupoïde différentiables*”.

**Definition 2.3** A *locally topological groupoid* is a pair  $(G, W)$  consisting of a groupoid  $G$  and a topological space smooth  $W$  such that:

- $G_1$ )  $O_G \subseteq W \subseteq G$ ;
- $G_2$ )  $W = W^{-1}$ ;

- $G_3$ ) the set  $W(\delta) = (W \times_\alpha W) \cap \delta^{-1}(W)$  is open in  $W \times_\alpha W$  and the restriction of  $\delta$  to  $W(\delta)$  is continuous;
- $G_4$ ) the restrictions to  $W$  of the source and target maps  $\alpha$  and  $\beta$  are continuous and the triple  $(\alpha, \beta, W)$  is locally sectionable;
- $G_5$ )  $W$  generates  $G$  as a groupoid.

Note that, in this definition,  $G$  is a groupoid but does not need to have a topology. The locally Lie groupoid  $(G, W)$  is said to be *extendible* if there can be found a topology on  $G$  making it a topological groupoid and for which  $W$  is an open subset. In general  $(G, W)$  is not extendible, but there is a holonomy groupoid  $Hol(G, W)$  and a morphism  $\phi: (G, W) \rightarrow G$  of groupoids such that  $Hol(G, W)$  admits the structure of topological groupoid. The construction is given in detail in [1]. For an example of locally topological groupoid which is not extendible see [1].

**Theorem 2.4** (*Globalisability theorem*) *Let  $(G, W)$  be a locally topological groupoid. Then there is a topological groupoid  $H$ , a morphism  $\phi: H \rightarrow G$  of groupoids and an embedding  $i: W \rightarrow H$  of  $W$  to an open neighbourhood of  $O_H$  such that the following conditions are satisfied.*

- i)  $\phi$  is the identity on objects,  $\phi i = id_W$ ,  $\phi^{-1}(W)$  is open in  $H$ , and the restriction  $\phi_W: \phi^{-1}(W) \rightarrow W$  of  $\phi$  is continuous;*
- ii) if  $A$  is a topological groupoid and  $\xi: A \rightarrow G$  is a morphism of groupoids such that:*
- a)  $\xi$  is the identity on objects;*
  - b) the restriction  $\xi_W: \xi^{-1}(W) \rightarrow W$  of  $\xi$  is smooth and  $\xi^{-1}(W)$  is open in  $A$  and generates  $A$ ;*
  - c) the triple  $(\alpha_A, \beta_A, A)$  is locally sectionable,*

*then there is a unique morphism  $\xi': A \rightarrow H$  of topological groupoids such that  $\phi \xi' = \xi$  and  $\xi' a = i \xi a$  for  $a \in \xi^{-1}(W)$ .*

The groupoid  $H$  is called the *holonomy groupoid*  $Hol(G, W)$  of the locally topological groupoid  $(G, W)$ .

Let  $G$  be a topological groupoid on  $X$ . Then  $G$  is called *locally trivial* if for all  $x \in X$  there is an open set  $U$  containing  $x$  and a section  $s: U \rightarrow G_x$  of  $\beta$ . Thus  $\beta s = 1_U$  and for each  $y \in U$ ,  $\alpha(s(y)) = x$ , i.e.  $s(y): x \rightarrow y$  in  $G$ .

The following result, whose proof is due to Pradines, is given in [1]. In the proof of this result the sectionable condition is used.

**Proposition 2.5** ( 6.1 in [1]) *A locally trivial locally topological groupoid is extendible.*

As a corollary of of Theorem 2.4 the following result is obtained. See [4] for a nice application of this result in the holonomy groupoids of local subgroupoids.

**Corollary 2.6** *Let  $G$  be a topological groupoid and let  $p: M \rightarrow G$  be a morphism of groupoids such that  $p: O_M \rightarrow O_G$  is the identity. Let  $W$  be an open subset of  $G$  such that*

- a)  $O_G \subseteq W$ ;*
- b)  $W = W^{-1}$ ;*
- c)  $W$  generates  $G$ ;*
- d)  $(\alpha_W, \beta_W, W)$  is locally sectionable;*

and suppose that  $\tilde{i} : W \rightarrow M$  is given such that  $\tilde{p} = i : W \rightarrow G$  is the inclusion and  $W' = \tilde{i}(W)$  generates  $M$ .

Then  $M$  admits a unique structure of topological groupoid such that  $W'$  is an open subset and  $p : M \rightarrow G$  is a morphism of topological groupoids mapping  $W'$  homeomorphically to  $W$ .

These imply Theorem 2.9 which is called the *Strong Monodromy Principle*, namely the globalisation of smooth local morphisms to smooth morphisms on the monodromy groupoid.

**Theorem 2.7** *Let  $G$  be a locally sectionable topological groupoid and let  $W$  be an open subset of  $G$  containing  $O_G$ , such that  $W = W^{-1}$ , and  $W$  generates  $G$ . Then the monodromy groupoid  $M = M(G, W)$  admits the structure of topological groupoid such that  $\tilde{i}(W)$  is an open subspace of  $M$  and any continuous pregroupoid morphism on  $W$  globalises to a continuous morphism on  $M$ .*

**Theorem 2.8** *Suppose further to the assumptions of Theorem 2.7 that  $G$  is star path-connected, that each of its stars has a simply connected covering, and that  $W^2$  is contained in an open neighbourhood  $V$  of  $O_G$  such that each star  $V_x$  in  $G_x$  is liftable. Then the projection  $p : M(G, W) \rightarrow G$  is the universal covering map on each star, and so  $M(G, W)$  is isomorphic to the star universal cover  $\Pi G$  of  $G$ .*

**Theorem 2.9** (Strong monodromy principle) *Let  $G$  be a locally sectionable star path-connected Lie groupoid and let  $W$  be a neighbourhood of  $O_G$  in  $G$  such that  $W$  satisfies the condition:*

( $\star$ )  *$W$  is path-connected and  $W^2$  is contained in a star path-connected neighbourhood  $V$  of  $O_G$  such that for all  $x \in O_G$ ,  $V_x$  is liftable.*

*Let  $f : W \rightarrow H$  be a continuous pregroupoid morphism from  $W$  to the topological groupoid  $H$ . Then  $f$  determines uniquely a morphism  $f' : \Pi G \rightarrow H$  of Lie groupoids such that  $f'j' = f$ .  $\square$*

### 3 Extendibility of compatible locally trivial groupoids

The concept of locally triviality is due to Ehresmann [7]. It is also used by Mackenzie in [10]. We adapt this concept as follows.

**Definition 3.1** Let  $X$  be a topological space and let  $G$  be a groupoid on  $X$ . Let  $\mathcal{U} = \{U_i : i \in I\}$  be an open cover of  $X$ , which is also a base for  $X$  such that if  $x \in U_i$  then there is a local section  $s_{x,i} : U_i \rightarrow G_x$  of  $\beta$  such that  $s_{x,i}(x) = x$ . Thus for  $y \in U_{x,i}$ ,  $s_{x,i}(y) : x \rightarrow y$  in  $G$ . Such a map  $s_{x,i} : U_i \rightarrow G_x$  is called *local section* about  $x \in O_G$ .  $G$  is called *compatible locally trivial* if the following compatible condition is satisfied:

**Comp:** For  $x \in X$  if  $s_{x,i} : U_i \rightarrow G_x$  and  $s_{x,j} : U_j \rightarrow G_x$  are two local sections about  $x$  then there is an open neighbourhood  $V_{ij}$  of  $x$  in  $\mathcal{U}$  such that  $V_{ij} \subseteq U_i \cap U_j$  and  $s_{x,i}|_{V_{ij}} = s_{x,j}|_{V_{ij}}$ .

If  $s_{x,i} : U_i \rightarrow G_x$  is a local section about  $x$ , then we write  $\tilde{U}_{x,i}$  for the image  $s_{x,i}(U_i)$ . We now prove the following theorem on the extendibility of compatible locally trivial groupoid to a topological groupoid. This result is similar to the fundamental groupoid case studied earlier in [3].

**Theorem 3.2** *Let  $G$  be a compatible locally trivial groupoid. Let  $a \in G(x, y)$ . Then the sets  $(\tilde{U}_{x,i})^{-1}a(\tilde{U}_{y,j})$  for all  $x \in U_i$  and  $y \in U_j$  form a set of basic neighbourhoods for a topology on  $G$  such that  $G$  is a topological groupoid with this topology.*

**Proof:** By the compatibility condition it is obvious that these sets form a topology on  $G$ . Because if  $(\tilde{U}_{x,i})^{-1}a(\tilde{U}_{y,j})$  and  $(\tilde{U}_{x,i'})^{-1}a(\tilde{U}_{y,j'})$  are basic open neighbourhood of  $a$ , then  $x \in U_i \cap U_{i'}$  and  $y \in U_j \cap U_{j'}$ . By the compatible conditions there is a basic open neighbourhood  $V_{ii'}$  of  $x$  in  $\mathcal{U}$  such that  $V_{ii'} \subseteq U_i \cap U_{i'}$  and  $s_{x,i}|_{V_{ii'}} = s_{x,i'}|_{V_{ii'}}$ . Similarly here is a basic open neighbourhood  $V_{jj'}$  of  $y$  such that  $V_{jj'} \subseteq U_j \cap U_{j'}$  and  $s_{x,j}|_{V_{jj'}} = s_{x,j'}|_{V_{jj'}}$ . Hence  $(\tilde{V}_{x,ii'})^{-1}a(\tilde{V}_{y,jj'})$  is a basic open neighbourhood of  $a$  such that

$$(\tilde{V}_{x,ii'})^{-1}a(\tilde{V}_{y,jj'}) \subseteq (\tilde{U}_{x,i})^{-1}a(\tilde{U}_{y,j}) \cap (\tilde{U}_{x,i'})^{-1}a(\tilde{U}_{y,j'})$$

We now prove that  $G$  is a topological groupoid with this topology. To prove that the groupoid difference map

$$\delta: G \times_{\alpha} G \rightarrow G, (a, b) \mapsto a^{-1}b$$

is continuous let  $\delta(a, b) = a^{-1}b$ , where  $a \in G(x, y)$  and  $b \in G(x, z)$ . Let  $(\tilde{U}_{y,j})^{-1}a^{-1}b(\tilde{U}_{z,k})$  be a basic neighbourhood of  $a^{-1}b$ . Then  $(\tilde{U}_{x,i})^{-1}a(\tilde{U}_{y,j}) \times_{\alpha} (\tilde{U}_{x,i})^{-1}b(\tilde{U}_{z,k})$  is an open neighbourhood of  $(a, b)$  and

$$\delta((\tilde{U}_{x,i})^{-1}a(\tilde{U}_{y,j}) \times_{\alpha} (\tilde{U}_{x,i})^{-1}b(\tilde{U}_{z,k})) = (\tilde{U}_{y,j})^{-1}a^{-1}b(\tilde{U}_{z,k}).$$

So  $\delta$  is continuous.

To prove the continuity of the target point map  $\beta: G \rightarrow X$  let  $a \in G$  with  $\alpha(a) = x$  and  $\beta(a) = y$  and let  $U_j$  be a basic open neighbourhood of  $y$  in  $\mathcal{U}$ . Then  $(\tilde{U}_{x,i})^{-1}a(\tilde{U}_{y,j})$  is a basic open neighbourhood of  $a$  and  $\beta((\tilde{U}_{x,i})^{-1}a(\tilde{U}_{y,j})) \subseteq U_j$ . Hence  $\beta$  is continuous. Further since  $\delta$  and  $\beta$  are both continuous so also is the source point map  $\alpha: G \rightarrow X$ .

Finally for the continuity of the identity point map  $\epsilon: X \rightarrow G$  let  $x \in X$  and let  $(\tilde{U}_{x,i})^{-1}1_x(\tilde{U}_{x,j})$  be a basic open neighbourhood of  $1_x$ . Then  $x \in U_i \cap U_j$  and by the compatibility condition there is an open neighbourhood of  $x$  in  $\mathcal{U}$  such that  $V_{ij} \subseteq U_i \cap U_j$  and  $s_{x,i}|_{V_{ij}} = s_{x,j}|_{V_{ij}}$ . So  $\epsilon(V_{ij}) \subseteq (\tilde{U}_{x,i})^{-1}1_x(\tilde{U}_{x,j})$ . Hence  $\epsilon$  is continuous.  $\square$

In particular in this result if we take  $G$  to be the fundamental groupoid  $\pi_1 X$  on a topological space  $X$  which is locally nice then we obtain a result given in [3] stated as follows:

**Example 3.3** Let  $X$  be a locally path connected and semilocally 1-connected topological space. Then there is an open cover  $\mathcal{U}$  of  $X$  as in Definition 3.1 consisting of all open, path connected subsets  $U$  of  $X$  such that there is only one homotopy class of the paths in  $U$  between same points. If  $x \in U_i$  for  $U_i \in \mathcal{U}$ , then the local section  $s_{x,i}: U_i \rightarrow \pi_1 X$  is defined by choosing for each  $y \in U_i$  a path in  $U_i$  from  $x$  to  $y$  and taking  $s_{x,i}(y)$  to be the homotopy class of this path. Since the path class of the paths in  $U_i$  between same points is unique, the local section  $s_{x,i}$  is well defined. Further since  $X$  is locally path connected the compatibility condition is satisfied. Thus  $\pi_1 X$  is a compatible locally trivial groupoid.

Let  $G$  be a topological groupoid and  $W$  an open neighbourhood of  $O_G$  such that  $O_G \subseteq W$ . Then we have the monodromy groupoid  $M(G, W)$  defined as in Definition 1.2. Note that by the construction of  $M(G, W)$  we have a pregroupoid morphism  $\tilde{\iota}: W \rightarrow M(G, W)$  which is an inclusion map. Let  $\widetilde{W}$  be the image of  $W$  under this inclusion and let  $\widetilde{W}$  have the topology such that  $\tilde{\iota}: W \rightarrow \widetilde{W}$  is a homeomorphism. Then we have following result.

**Proposition 3.4** *Let  $G$  be a topological groupoid whose underlying groupoid is compatible locally trivial and let  $W$  be an open subset of  $G$  such that  $O_G \subseteq W$ . Suppose that each local section  $s_{x,i}: U_{x,i} \rightarrow G_x$  has image in  $W$ , i.e.  $s_{x,i}(U_{x,i}) \subseteq W$ . Then the groupoid  $M(G, W)$  is compatible locally trivial.*

**Proof:** The proof is immediate since  $W$  is isomorphic to  $\widetilde{W}$  and the inclusion  $\tilde{\iota}: W \rightarrow \widetilde{W}$  is identity on objects.  $\square$

As a corollary of this result we state the following.

**Corollary 3.5** *Let  $G$  be a topological groupoid whose groupoid is compatible locally trivial and let  $W$  be an open subset of  $G$  including all the identities. Suppose that each local section  $s_{x,i}: U_{x,i} \rightarrow G_x$  has image in  $W$ . Then the monodromy groupoid  $M(G, W)$  has a topology turning it into a topological groupoid.*

No now give a result which is similar to Theorem 3.2 under strong conditions.

**Theorem 3.6** *Let  $X$  be a topological space and  $G$  groupoid on  $X$ . Let  $W$  be a compatible locally trivial subgroupoid of  $G$ . Then  $G$  has a topological groupoid making it a topological groupoid such that  $W$  is open in  $G$ .*

**Proof:** By Theorem 3.2,  $G$  has a topology making it a topological groupoid. Let  $a \in W$  with  $\alpha(a) = x$  and  $\beta(b) = y$ . Since for  $x \in O_G$ ,  $s_{x,i}(x) = 1_x$ ,  $(\tilde{U}_{x,i})^{-1}a(\tilde{U}_{y,j})$  is an open neighbourhood of  $a$  and since  $W$  is a subgroupoid,  $(\tilde{U}_{x,i})^{-1}a(\tilde{U}_{y,j}) \subseteq W$ . So  $W$  is open in  $G$ .  $\square$

**Corollary 3.7** *Let  $G$  be a topological groupoid and let  $W$  be a compatible locally trivial subgroupoid of  $G$ . Then the monodromy groupoid  $M(G, W)$  has a topology turning into a topological groupoid such that  $\tilde{\iota}(W) = \widetilde{W}$  is open in  $M(G, W)$  and any continuous pregroupoid morphism  $f: W \rightarrow H$  globalises to a morphism of topological groupoids  $\tilde{f}: M(G, W) \rightarrow H$ .*

**Proof:** Since  $\tilde{\iota}: W \rightarrow \widetilde{W}$  is a homeomorphism and  $O_G = O_{M(G, W)}$ ,  $\widetilde{W}$  is a compatible locally trivial subgroupoid of  $M(G, W)$ . So by Theorem 3.6,  $M(G, W)$  becomes a topological groupoid such that  $\widetilde{W}$  is open in  $M(G, W)$ . Let  $f: W \rightarrow H$  be a continuous pregroupoid morphism. By the universal property of  $M(G, W)$ ,  $f$  globalises to a morphism  $\tilde{f}: M(G, W) \rightarrow H$  which is continuous on  $\widetilde{W}$ . But  $\widetilde{W}$  is open in  $M(G, W)$  and  $\widetilde{W}$  generates  $M(G, W)$ . So the continuity of  $\tilde{f}$  on whole  $M(G, W)$  follows.  $\square$

**Theorem 3.8** *Let  $G$  be a topological groupoid which is compatible locally trivial such that each star  $G_x$  has a universal covering and let  $W$  be an open subset of  $G$  such that  $O_G \subseteq W$  and  $W^2$  is contained in an open neighbourhood  $V$  of  $O_G$  such that each star  $V_x$  is liftable. Suppose that each local section  $s_{x,i}: U_{x,i} \rightarrow G_x$  has image in  $W$ . Then the monodromy groupoid  $M(G, W)$  may be given a topological groupoid structure such that each star  $M(G, W)_x$  is isomorphic to the universal covering of  $G_x$ .*



**Proof:** By Proposition 3.4, the groupoid  $M(G, W)$  is compatible locally trivial and by Theorem 3.2 it becomes a topological groupoid. Further by Theorem 2.1 we identify  $M(G, W)$  with  $\Pi G$ , the union of all the universal coverings of the stars  $G_x$ 's.  $\square$

As a corollary of these results we give the following.

**Theorem 3.9** (monodromy principle) *Let  $G$  be a topological groupoid in which each star  $G_x$  is path connected and has a universal covering and let  $W$  be a locally compatible subgroupoid of  $O_G$  such that  $W$  is star liftable. Then any continuous pregroupoid morphism  $f : W \rightarrow H$  determines uniquely a morphism  $f' : \Pi G \rightarrow H$  of topological groupoids such that  $f'j = f$ .*  $\square$

**Acknowledgement:** We would like to thank Prof. Ronald Brown for introducing this area to us and his help and encouragement.

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